MATC44 Week 1 Notes

1. Combinatorial Principles:
   1. **Contradiction**: With contradiction, we assume that what we want to show does not hold and use this extra assumption to reach an obviously wrong conclusion. This is one of the most powerful techniques in math because it allows for the introduction of one additional assumption, the contradiction assumption, which can be freely used.

   2. **Reduction**: Address the problem by first lowering its complexity.
      E.g. By reducing the number of variables.

   3. **Induction**: Solve the general problem by using the solution for the reduced problem.

E.g. 1 The prisoner's problem

Consider 100 prisoners on an island s.t.

1. They all have green eyes.
2. They can all see each other, but can't see themself.
3. No communication is allowed.
4. They don't know their own eye color.
5. Any prisoner can go to the guards at any midnight and claim they have green eyes. If the prisoner does have green eyes, they are freed. Otherwise, they are killed.

One day, the warden tells all the prisoners: "At least one of you have green eyes."

Prove how the prisoners can all escape.

Soln:
Lets use reduction and reduce the number of prisoners to 2, A and B. All other conditions remain the same. Suppose we are A. Lets use contradiction and assume we (A) does NOT have green eyes. Then:

- We know that B can see us. Therefore, B knows that we don't have green eyes.
- B knows that at least one person has green eyes. Since B knows that doesn't have green eyes, by default, B knows he must have green eyes.
- We expect B to escape and be gone by the next day. However, in reality, because A does have green eyes, B didn't escape.
- When A sees B the next day, A knows that he must have green eyes and escapes.

We can use the same thought process, but from B's perspective to help B escape.

$\therefore$ Both A and B both escape on the 2nd day.

Now, suppose we have 3 prisoners, A, B and C. Suppose we are A. Lets use contradiction and assume we don't have green eyes. Then:

- We know that both B and C knows we don't have green eyes.
- We know that both B and C will escape on the second day. However, in reality, B and C are still in the prison on the third day.
- We know that we have green eyes and escape on the third day.

The same thought process is used by B and C, but from their perspective, to escape.

∴ All 3 prisoners escape on the third day.

The case with 100 prisoners is solved inductively. Each prisoner assumes that they do not have green eyes, and therefore the problem is reduced to the case of 99 prisoners. By induction, the 99 prisoners should escape on the 99^{th} day, but this doesn't happen. Therefore, on the 100^{th} day, all prisoners realize they have green eyes and escape at midnight.

2. Pigeon Hole Principle (P.P):

- Version 1: If we place $(n+1)$ pigeons in n pigeonholes, then there must be at least 1 pigeonhole with more than 1 pigeons.

  Proof: Suppose, by contradiction, that each pigeonhole (ph) contains at most 1 pigeon. Then, we would have at most n pigeons, which contradicts our statement.
  $\therefore$ There must be at least 1 ph with more than 1 pigeon.

- Version 2: If we place $n \cdot m + 1$ pigeons in n phs, then there must be at least $(m+1)$ pigeons in the same ph.

  Proof: By contradiction, assume that each ph has at most m pigeons. Then, there are at most $n \cdot m$ pigeons in total, which contradicts our earlier claim.
  $\therefore$ There must be at least $(m+1)$ pigeons in the same ph.

- E.g. Prove that in any room of 13 people, there are always 2 born in the same month.

  Soln:
  In this example, the 13 ppl are the pigeons and the number of months per year (12) are like the phs. Since there are $n+1$ pigeons in n pigeonholes, there must be at least 1 ph with more than 1 pigeon.

By the P.P, in any group of 13 ppl,
I.e. there must be at least 2 people born in the same month.

- E.g. Prove that among 37 people, there are always 4 ppl born in the same month.

Soln:

In this example, the 37 ppl are the pigeons and the number of months in a year is the ph.
$37 = 3 \cdot 12 + 1$
$\therefore$ By the P.P, among 37 people, there are always 4 ppl born in the same month.

- E.g. Suppose we have 4 phs. What is the least amount of pigeons needed s.t. at least one ph has at least 3 pigeons?

Soln:
Recall version 2: "If we place $n \cdot m + 1$ pigeons in n phs, then there must be at least $(m+1)$ pigeons in the same ph."
In our case:
$n = 4$
$m+1 = 3 \rightarrow m = 2$
$n \cdot m + 1 = 9$
$\therefore$ 9 pigeons is the min number of pigeons needed s.t. at least one ph has 3 pigeons.

- Systematic Approach: Let v be the num of vars in our problem. If we want to show that at least ℓ of them have property P, then:
  1. v is the number of pigeons.
     $v = n \cdot m + 1$
  2. $\ell = m+1$
  3. We must have at most n properties.
     $$n = \frac{v-1}{m}$$
     $$= \frac{v-1}{\ell-1}$$

- E.g. 1 Show that if we randomly select 4 nums from the set $A = \{1, 2, 3, \dots, 16, 17, 18\}$, then there must always be 2 of them, x and y, s.t. $|x-y| \leq 5$.

Soln:
1. Identify what the pigeon is. In this case, the pigeon is the 4 numbers chosen.

2. Identify what the ph is. If we split A into the following sets, $\{1, 2, 3, 4, 5, 6\}$, $\{7, 8, 9, 10, 11, 12\}$ and $\{13, 14, 15, 16, 17, 18\}$, the new sets are the phs. Note that in each of the phs, the absolute difference of the smallest and largest values is 5. By the P.P, there must be at least 2 numbers from the same set.

$\therefore$ There must always be 2 numbers x and y s.t. $|x-y| \leq 5$.

Note: The way we split A is important. We had to split it s.t. the abs diff of the largest and smallest values in each set is 5.

- E.g. 2. In any collection of $(n+1)$ items from the set $A = \{1, 2, \dots, 2n\}$, show that there are 2 of them that are consecutive.

Soln:

1. Identify the pigeons. In this case, the pigeons are the $(n+1)$ items.

2. Identify the ph. Lets split A into n subsets of consecutive values.
   I.e. $A = \{1,2\} \cup \{3,4\} \dots \cup \{2n-1, 2n\}$.
   These subsets are the phs.
   By the P.P, 2 of the $(n+1)$ items must be in the same subset, and as such, are consecutive.

- E.g. 3. We are given 33 ppl s.t. in any group of 9, there will always be 2 of them that have the same height. Prove that among the 33 ppl, there are 5 ppl with the same height.

Soln:

Following the systematic approach, $V = 33$ and $\ell = 5$.

$$n = \frac{V-1}{\ell-1} = \frac{33-1}{5-1} = \frac{32}{4} = 8$$

Based on the condition that in any group of 9 ppl, there will always be 2 of them with the same height, that means there are at most 8 distinct heights. By the P.P, one of the 8 distinct heights will be shared by 5 ppl. I.e. By the P.P, 5 ppl will have one of the 8 distinct heights.

- E.g. 4 Given any $(n+1)$ natural numbers, there are always 2 numbers, x and y, s.t. $x-y$ is a multiple of n.

Soln:

Lemma:

Let $x, y \in \mathbb{N}$

$x-y$ = multiple of n iff $x \equiv y \pmod{n}$

We want to show that there are 2 numbers that have the same remainder when divided by n.

Since we are by n, there are n possible types of remainders: $\{0, 1, 2, \ldots, n-1\}$.

By the P.P, 2 of the numbers from the $(n+1)$ numbers must have the same remainder when divided by n and, as such, their difference is a multiple of n.

– E.g. 5 Among n consecutive natural numbers, there is always one of them which is divisible by n.

Soln:

Lemma: The difference of any 2 numbers in a set of n consecutive numbers is always between 1 and $n-1$.

We want to show that the remainder of one of the n natural numbers divided by n is 0.

By contradiction, assume that none of the n natural numbers get a remainder of 0 when divided by n.
I.e. The set of possible remainders is $\{1, 2, \ldots, n-1\}$.
There are $n-1$ remainders and n numbers.
By the P.P, there must be 2 numbers that have the same remainder. From example 4, we proved that the diff of the 2 numbers is 0.
However, this contradicts our lemma statement.
$\therefore$ One of the n natural numbers must get a remainder of 0 when divided by n.

- E.g. 6 Consider the real numbers $a_1, a_2, \ldots, a_n$ s.t. $0 \leq a_i \leq n$, for all $i = 1, 2, \ldots, n$. Show that there are 2 distinct a_i, a_j s.t. $|a_i - a_j| \leq \frac{n}{n-1}$.

Soln:
Lets split $[0, n]$ into the following intervals:
$0, \frac{n}{n-1}, \frac{2n}{n-1}, \frac{3n}{n-1}, \ldots, \frac{(n-2)(n)}{n-1}, n.$
Note that the difference between these intervals is $\frac{n}{n-1}$. Furthermore, there are $n-1$ intervals. By the P.P, there must be at least 2 numbers that are in the same interval, and as such, their abs diff is less than or equal to $\frac{n}{n-1}$.

- E.g. 7 Consider 6 mutually distinct numbers $X_i \in \{1, 2, \ldots, 9\}$, $i = 1, 2, \ldots, 6$. Show that there are 4 $X_{i1}, X_{i2}, X_{i3}, X_{i4}$ of these numbers s.t. $X_{i1} + X_{i2} = X_{i3} + X_{i4}$.

Soln:
Let A, B, C, D, E and F be our 6 numbers. There are 15 distinct sums that can be made with these 6 numbers:

A+B	B+C	C+D	D+E	E+F
A+C	B+D	C+E	D+F	
A+D	B+E	C+F		
A+E	B+F			
A+F				

Furthermore, there are 15 distinct possible sums that can be made by any 2 numbers from the set $\{1, 2, \ldots, 9\}$. These sums are $\{3, 4, \ldots, 16, 17\}$. Note that 3 and 17 only have 1 way of being created: $1+2$ and $8+9$, respectively. Let's split this into 3 cases:

Case 1: 3 is not attained. If 3 is not attained, then there are 14 distinct, possible sums. By the P.P, there must be a case s.t. $a+b = c+d$.

Case 2: 17 is not attained. Just like case 1, there are 14 distinct, possible sums. By the P.P, there must be a case s.t. $a+b=c+d$.

Case 3: Both 3 and 17 are attained.

$$\left.\begin{array}{l} 3=1+2 \\ 17=8+9 \end{array}\right\} 1+9=2+8$$

— E.g. 8. Show that in any collection of $n+1$ numbers from the set $\{1, 2, \ldots, 2n\}$, there are 2 s.t. one is a multiple of the other. n is a natural number.

Soln:

Every natural num, m, can be written as $m = 2^k \cdot o$, where o is an odd number known as the greatest odd divisor of m.

For each odd number, $2x-1$, in the set $\{1, 2, \ldots, 2n\}$, we will make the set S_x s.t. $S_x = \{2x-1, 2(2x-1), 4(2x-1), 2^k(2x-1)\}$.
I.e. S_x contains the number $2x-1$ and any number that can be obtained by multiplying $2x-1$ with a power of 2.

E.g. Suppose $n=4$

$S_1 = \{1, 2, 4, 8\}$
$S_2 = \{3, 6\}$
$S_3 = \{5\}$
$S_4 = \{7\}$

E.g. Suppose $n=7$
$S_1 = \{1, 2, 4, 8\}$
$S_2 = \{3, 6, \quad, 12\}$
$S_3 = \{5, 10\}$
$S_4 = \{ \quad 7, 14\}$
$S_5 = \{9\}$
$S_6 = 11$
$S_7 = 13$

Because there are n odd numbers in the set $\{1, 2, \ldots, 2n\}$, there will be n S_x sets. Since we are selecting $(n+1)$ numbers, by the P.P, there will always be 2 numbers in the same S_x set. ∴ There will always be 2 numbers s.t. one is a multiple of the other.